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A CONSUMPTION, PORTFOLIO AND RETIREMENT CHOICE PROBLEM WITH NEGATIVE WEALTH CONSTRAINTS

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Abstract. In this paper we study an optimal consumption, investment and retirement time choice problem of an investor who receives labor income before her voluntary retirement. And we assume that there is a negative wealth constraint which is a general version of borrowing constraint. Using convex-duality method, we provide the closed-form solutions of the optimization problem.

1. Introduction

We investigate an optimal consumption, portfolio and retirement time choice problem of an investor with negative wealth constraints. We assume that an investor receives a labor income and suffers disutility until the voluntary retirement time.

A voluntary retirement time problem is a meaningful extension of the optimal consumption and portfolio selection problem. And a voluntary retirement time problem can be formulated as a kind of an optimal stopping problem. So many researchers have studied and have obtained important economic results. (see [4], [3], [1], [2])

Roh [7] studied the optimal portfolio and consumption selection problem, but he did not consider voluntary retirement. Lee et al. [5] investigated a model of optimal consumption, portfolio and voluntary retirement time with nonnegative wealth constraints and they used the dynamic programming method. We extend this model to a negative wealth constraint and we use a convex-duality method which is suggested in Park and Jang [6] for solving our model. Park and Jang [6]

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studied an optimal consumption and portfolio selection problem of an investor who has a Cobb-Douglas utility function.

2. The model

We assume that there is a filtered probablility space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$. The filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfies the usual conditions and is generated by one-dimensional Brownian motion B_t. We assume that there are two assets which are able to be invested in the financial market. And we suppose that the return of riskless asset evolves according to $dP_t = rP_t dt$, and the price process of risky asset evolves according to the form,

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{2.1}$$

We defined $X_t = \left\{ rX_t + \pi_t(\mu - r) - c_t + \epsilon 1_{\{0\leq t<\tau\}} \right\} dt + \sigma \pi_t dB_t$ as a wealth process of agent. Here, c_t is the consumption rate at time t, π_t is the amount of money invested in the risky asset at time t, and ϵ is a constant labor income rate. And we define that τ is a voluntary retirement time. Then the agent receives the labor income before retirement time τ.

We assume that there are negative wealth constraints as follows:

$$X_t \geq -\nu \frac{\epsilon}{r} 1_{\{t<\tau\}}, \ \nu \in [0, 1].$$

Here $\nu = 0$ implies a nonnegative wealth constraint. And if $\nu = 1$, then the investor is able to borrow the discounted sum of future labor income.

3. The optimization problem

The investor's optimization problem is to maximize her expected utility by choosing consumption, portfolio, and retirement time:

$$\begin{aligned} V(x) &= \max_{c,\pi,\tau} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} - l 1_{\{0\leq t<\tau\}} \right) dt \right] \\ &= \max_{c,\pi,\tau} \mathbb{E}\left[\int_0^\tau e^{-\rho t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} - l \right) dt + \int_\tau^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right], \end{aligned} \tag{3.1}$$

where $\gamma > 0 (\gamma \neq 1)$ is an investor's coefficient of relative risk aversion, $\rho > 0$ is a subjective discount rate, and $l > 0$ is a constant disutility due to labor. Moreover we define the Merton's constant $K > 0$ such that

$$K := r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2.$$

REMARK 3.1. We define a quadratic equation,

$$g(m) = \frac{1}{2}\theta^2 m^2 + \left(\rho - r + \frac{1}{2}\theta^2\right) m - r = 0,$$

with two roots $m_+ > 0$ and $m_- < -1$.

ASSUMPTION 3.2. We assume that the following inequalities are satisfied.

$$(1-\nu)\frac{\epsilon}{r} > \frac{1}{K}\left(1 + \frac{1}{\gamma m_-}\right)\hat{\lambda}^{-\frac{1}{\gamma}},$$
$$g(1) = \theta^2 - 2r + \rho \geq 0.$$

LEMMA 3.3.

$$V_1(x) = \max_{c,\pi} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] = \frac{1}{1-\gamma}\frac{1}{K^\gamma}x^{1-\gamma}.$$

Since $V_1(x)$ is the value of the classical consumption and portfolio Merton problem, we omit the proof of Lemma 3.3.

Using Lemma 3.3, the value function (3.2) is converted as follows:

$$V(x) = \max_{c,\pi,\tau} \mathbb{E}\left[\int_0^\tau e^{-\rho t}\left(\frac{c_t^{1-\gamma}}{1-\gamma} - l\right)dt + e^{-\rho\tau}V_1(X_\tau)\right]. \tag{3.2}$$

THEOREM 3.4. *The value function is given as following*

$$V(x) = \begin{cases} \frac{m_+}{m_++1}C_1\lambda^{m_++1} + \frac{m_-}{m_-+1}C_2\lambda^{m_-+1} + \frac{1}{K(1-\gamma)}\lambda^{1-1/\gamma} - \frac{l}{\rho} & \text{if } x \in [-\nu\frac{\epsilon}{r}, \bar{x}), \\ \frac{1}{1-\gamma}\frac{1}{K^\gamma}x^{1-\gamma} & \text{if } x \in [\bar{x}, \infty). \end{cases}$$

Here m_+, m_- *are two roots of Remark 3.1 and* $\xi \in (0,1)$ *is a solution of the following equation,*

(3.3)

$$\begin{aligned} &2l\left[\frac{\gamma\epsilon K}{r}\frac{(1-\nu)(m_+\xi^{m_-} - m_-\xi^{m_+}) - m_+ + m_-}{\xi^{m_+}(1+\gamma m_-) - \xi^{m_-}(1+\gamma m_+)}\right]^\gamma \\ &+ \frac{1+\gamma m_+}{r}\left(\frac{(1-\nu)(m_+\xi^{m_-} - m_-\xi^{m_+}) - m_+ + m_-}{\xi^{m_+}(1+\gamma m_-) - \xi^{m_-}(1+\gamma m_+)}\right)\epsilon\theta^2\xi^{1+m_-} \\ &- 2\epsilon\xi + \frac{\theta^2\epsilon m_+\xi}{r}\left\{1 - (1-\nu)\xi^{m_-}\right\} = 0. \end{aligned}$$

If we define

$$\hat{\lambda} = \left[\frac{\gamma\epsilon K}{r}\frac{(1-\nu)(m_+\xi^{m_-} - m_-\xi^{m_+}) - m_+ + m_-}{\xi^{m_+}(1+\gamma m_-) - \xi^{m_-}(1+\gamma m_+)}\right]^{-\gamma}, \bar{\lambda} = \xi\hat{\lambda},$$

then the coefficients C_1, C_2 and the retirement wealth level $\bar{x}$ are given by

$$C_1 = \frac{-(1-\nu)\frac{\epsilon}{r}m_- + \frac{1}{K}(m_- + \frac{1}{\gamma})\hat{\lambda}^{-\frac{1}{\gamma}}}{m_+ - m_-}\hat{\lambda}^{-m_+}, \tag{3.4}$$

$$C_2 = \frac{(1-\nu)\frac{\epsilon}{r}m_+ - \frac{1}{K}(m_+ + \frac{1}{\gamma})\hat{\lambda}^{-\frac{1}{\gamma}}}{m_+ - m_-}\hat{\lambda}^{-m_-}, \tag{3.5}$$

$$\bar{x} = \frac{1}{K}\bar{\lambda}^{-\frac{1}{\gamma}}, \tag{3.6}$$

and λ is a solution to the algebraic equation

$$x = C_1\lambda^{m_+} + C_2\lambda^{m_-} + \frac{1}{K}\lambda^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}.$$

Proof. For $t \in [0, \tau)$, the Bellman equation of the optimization problem (3.2) is derived as follows :

$$\rho V(x) = \max_{c,\pi}\left[\{rx - c + \pi(\mu - r) + \epsilon\} V'(x) + \frac{1}{2}\pi^2\sigma^2 V''(x) + \frac{c^{1-\gamma}}{1-\gamma} - l\right]. \tag{3.7}$$

We put the FOCs of HJB equation (3.7) into equation (3.7), then we obtain the following equation,

$$(rx+\epsilon)V'(x) + \frac{\gamma}{1-\gamma}\left(V'(x)\right)^{1-1/\gamma} - \frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)} - \rho V(x) - l = 0. \tag{3.8}$$

Differentiating the equation (3.8) with respect to x, we can obtain the following equation :

$$\begin{aligned}(r-\rho)V'(x) + &(rx+\epsilon)V''(x) - (V'(x))^{-\frac{1}{\gamma}}V''(x) \\ &- \frac{\theta^2}{2}\frac{2V'(x)(V''(x)^2 - (V'(x)^2 V'''(x)}{(V''(x))^2} = 0.\end{aligned}$$

If we define that $\lambda(x) = V'(x)$ and $G(\lambda(x)) = x + \frac{\epsilon}{r}$, then we get the following ordinary differential equation(ODE) :

$$\frac{1}{2}\theta^2\lambda^2 G''(\lambda) + (\rho - r + \theta^2)\lambda G'(\lambda) - rG(\lambda) + \lambda^{-\frac{1}{\gamma}} = 0. \tag{3.9}$$

A general solution to the second-order ODE (3.9) is given as

$$G(\lambda) = C_1\lambda^{m_+} + C_2\lambda^{m_-} + \frac{1}{K}\lambda^{-\frac{1}{\gamma}},$$

where C_1 and C_2 are constants to be determined, and m_+ and m_- are solutions to be mentioned in Remark 3.1. Let

$$H(\lambda) = \frac{1}{\rho}\left\{r\lambda G(\lambda) - \frac{1}{2}\theta^2\lambda^2 G'(\lambda) + \frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} - l\right\},$$

then we have $V(x) = H(\lambda(x))$. Thus we derive the value function

(3.10)

$$V(x) = \frac{m_+}{m_+ + 1}C_1\lambda^{m_+ + 1} + \frac{m_-}{m_- + 1}C_2\lambda^{m_- + 1} + \frac{1}{K(1-\gamma)}\lambda^{1-\frac{1}{\gamma}} - \frac{l}{\rho},$$

where λ satisfies the algebraic equation

$$x = C_1\lambda^{m_+} + C_2\lambda^{m_-} + \frac{1}{K}\lambda^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}. \tag{3.11}$$

For $t \in [\tau, \infty)$, the optimization problem (3.2) is equal to the classical Merton problem. So the value function $V(x) = V_1(x)$ to be in Lemma 3.3.

Now we consider the boundary conditions of the optimization problem. From the negative wealth constraint, we obtain the following relations,

$$X(\hat{\lambda}) = -\nu\frac{\epsilon}{r}, X'(\hat{\lambda}) = 0, \tag{3.12}$$

where the wealth $X(\lambda)$ is given in (3.11) and $\hat{\lambda}$ is the convex dual variable corresponding to the negative wealth constraint $X(\hat{\lambda}) = \hat{x} = -\nu\frac{\epsilon}{r}$. From the equations (3.12), we get the coefficients as follows:

$$C_1 = \frac{1}{m_+ - m_-}\left\{-(1-\nu)\gamma\frac{\epsilon}{r}m_- + \frac{1}{K}(1+\gamma m_-)\hat{\lambda}^{-\frac{1}{\gamma}}\right\}\hat{\lambda}^{-m_+} \tag{3.13}$$

and

$$C_2 = \frac{1}{m_+ - m_-}\left\{(1-\nu)\gamma\frac{\epsilon}{r}m_+ - \frac{1}{K}(1+\gamma m_+)\hat{\lambda}^{-\frac{1}{\gamma}}\right\}\hat{\lambda}^{-m_-}. \tag{3.14}$$

We define that $\bar{\lambda}$ is the convex dual variable corresponding to the threshold wealth level for voluntary retirement. Then there is $\xi \in (0,1)$ such that $\bar{\lambda} = \xi\hat{\lambda}$.

By the smooth-pasting condition at $x = \bar{x}$, we obtain the following equations:

$$\begin{aligned} V(\bar{x}) &= \frac{1}{1-\gamma}\frac{1}{K^\gamma}\bar{x}^{1-\gamma}, \\ V'(\bar{x}) &= \bar{\lambda} = \frac{1}{K^\gamma}\bar{x}^{-\gamma} \end{aligned} \tag{3.15}$$

From the second equation in (3.15), we get the relation

$$\bar{x} = \frac{1}{K}\bar{\lambda}^{-\frac{1}{\gamma}}.$$

Put this relation into the first equation in (3.15) and (3.10), we obtain the equation as follows:

$$\frac{m_+}{m_+ + 1}C_1\bar{\lambda}^{m_+ + 1} + \frac{m_-}{m_- + 1}C_2\bar{\lambda}^{m_- + 1} = \frac{l}{\rho}. \tag{3.16}$$

Using the relation $\bar{\lambda} = \xi\hat{\lambda}$, the equation (3.16) is rewritten as follows:

$$\frac{m_+}{m_+ + 1}C_1\xi^{m_+ + 1}\hat{\lambda}^{m_+ + 1} + \frac{m_-}{m_- + 1}C_2\xi^{m_- + 1}\hat{\lambda}^{m_- + 1} = \frac{l}{\rho}. \tag{3.17}$$

Combining the equation (3.11) and the second equation in (3.15), we can obtain the following equation,

$$C_1\xi^{m_+}\hat{\lambda}^{m_+} + C_2\xi^{m_-}\hat{\lambda}^{m_-} = \frac{\epsilon}{r}. \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$C_1 = \left(\frac{l}{\rho} - \frac{\epsilon}{r}\frac{m_-}{m_- + 1}\xi\hat{\lambda}\right)\frac{-2\rho}{\theta^2(m_+ - m_-)}\xi^{-m_+ - 1}\hat{\lambda}^{-m_+ - 1} \tag{3.19}$$

and

$$C_2 = \left(\frac{l}{\rho} - \frac{\epsilon}{r}\frac{m_+}{m_+ + 1}\xi\hat{\lambda}\right)\frac{2\rho}{\theta^2(m_+ - m_-)}\xi^{-m_- - 1}\hat{\lambda}^{-m_- - 1}. \tag{3.20}$$

From (3.13) and (3.19), we have

(3.21)

$$(1-\nu)\frac{\epsilon}{r}m_+ - \frac{1}{K}(m_+ + \frac{1}{\gamma})\hat{\lambda}^{-\frac{1}{\gamma}} = \frac{2l}{\theta^2}\xi^{-m_- - 1}\hat{\lambda}^{-}1 - \frac{2\rho}{\theta^2}\frac{\epsilon}{r}\frac{m_+}{m_+ + 1}\xi^{-m_-}.$$

From (3.14) and (3.20), we also have

(3.22)

$$(1-\nu)\frac{\epsilon}{r}m_- - \frac{1}{K}(m_- + \frac{1}{\gamma})\hat{\lambda}^{-\frac{1}{\gamma}} = \frac{2l}{\theta^2}\xi^{-m_+ - 1}\hat{\lambda}^{-}1 - \frac{2\rho}{\theta^2}\frac{\epsilon}{r}\frac{m_-}{m_- + 1}\xi^{-m_+}.$$

From (3.21) and (3.22), we can derive

$$\hat{\lambda} = \left[\frac{\gamma\epsilon K}{r}\frac{(1-\nu)(m_+\xi^{m_-} - m_-\xi^{m_+}) - (m_+ - m_-)}{\xi^{m_+}(1+\gamma m_-) - \xi^{m_-}(1+\gamma m_+)}\right]^{-\gamma}. \tag{3.23}$$

If we put the equation (3.23) into (3.21), then we obtain the equation (3.3). □

LEMMA 3.5. *The coefficients C_1 and C_2 in Theorem 3.4 are positive.*

Proof. By the first inequality in Assumption 3.2, the equations (3.13) and (3.14), it is easy to prove that the coefficients C_1 and C_2 are positive. □

LEMMA 3.6. *$G(\lambda)$ is a decreasing function for $\lambda \in (\bar{\lambda}, \hat{\lambda})$.*

Proof. By the boundary condition of the negative wealth constraint, we obtain $G'(\hat{\lambda}) = 0$. It is enough to show that $G''(\lambda) > 0$ on $(\bar{\lambda}, \hat{\lambda})$. By the Lemma 3.6 and the second inequality in Assumption 3.2, we know that

$$G''(\lambda) = C_1 m_+(m_+ - 1)\lambda^{m_+-2} + C_2 m_-(m_- - 1)\lambda^{m_--2} + \frac{1}{\gamma K}\left(\frac{1+\gamma}{\gamma}\right)\lambda^{-\frac{1}{\gamma}-2} > 0.$$

Hence $G'(\lambda) < 0$ for $\lambda \in (\bar{\lambda}, \hat{\lambda})$. It means that $G(\lambda)$ is a decreasing function on $(\bar{\lambda}, \hat{\lambda})$. □

THEOREM 3.7. *The optimal consumption and portfolio strategy (c^*, π^*) is given as follows :*

$$c_t^* = \begin{cases} \zeta_t^{-\frac{1}{\gamma}} & \text{if } x \in [-\nu\frac{\epsilon}{r}, \bar{x}), \\ K x_t & \text{if } x \in [\bar{x}, \infty), \end{cases}$$

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma\gamma}\left(-C_1\gamma m_+\zeta_t^{m_+} - C_2\gamma m_-\zeta_t^{m_-} + \frac{1}{K}\zeta_t^{-\frac{1}{\gamma}}\right) & \text{if } x \in [-\nu\dfrac{\epsilon}{r}, \bar{x}), \\ \frac{\theta}{\gamma\sigma}x_t & \text{if } x \in [\bar{x}, \infty), \end{cases}$$

$$\theta = \inf\{t \geq 0 : X_\tau \geq \bar{x}\},$$

where ζ_t satisfies the following equation,

$$x_t + \frac{\epsilon}{r} = C_1\zeta_t^{m_+} + C_2\zeta_t^{m_-} + \frac{1}{K}\zeta_t^{-\frac{1}{\gamma}}.$$

References

[1] K. J. Choi and G. Shim, *Disutility, optimal retirement, and portfolio selection*, Math. Fin., **16** (2006), 443–467.

[2] K. J. Choi, G. Shim, and Y. H. Shin, *Optimal portfolio, consumption-leisure and retirement choice problem with CES utility*, Math. Fin., **18** (2008), 445–472.

[3] P. H. Dybvig and H. Liu, *Verification theorems for models of optimal consumption and investment with retirement and constrained borrowing*, Math. Oper. Res., **36** (2011), 620–635.

[4] E. Farhi and S. Panageas, *Saving and investing for early retirement: A theoretical analysis*, J. Financial Econ., **83** (2007), 87–121.

[5] H. -S. Lee, B. L. Koo, and Y. H. Shin, *A dynamic programming approach to a consumption/investment and retirement choice problem under borrowing constraints*, Japan J. Indust. Appl. Math., **34** (2017), 793–809.

[6] S. Park and B.-G. Jang, *Optimal retirement strategy with a negative wealth constraint*, Oper. Res. Lett., **42** (2014), 208–212.

[7] K. -H. Roh, *Portfolio and consumption optimization problem with Cobb-Douglas utility and negative wealth constraints*, J. Appl. Math. & Informatics, **36** (2018), 301–306.

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